So Fourier transform is a step forward
from what we learned in the last lecture.
So Fourier series, now we have transform.
So what's the difference?
And I will explain.
But the prerequisite is that you have a good understanding
of Fourier series.
Again, for those who came in later,
and if you feel really lost in Fourier series,
stay in the room after the lecture.
We chat a little bit.
And it sounds to me majority students are on track.
And you may not understand everything.
But you need to review and do homework.
And you check the chapter I wrote.
You check Google using multiple ways
and in group discussions.
Fourier analysis is so important.
I don't want you to get lost now.
Otherwise, you will not enjoy medical imaging modalities
later on.
Anyway, so this is our schedule.
We are on schedule, so no problem.
And the basic idea, let me reduce
to some very fundamental basic idea.
You have a general function.
The function you know is, say, in one-dimensional case,
f of t, that's a function of t.
You think it's a time-varying function.
And you can view function as it is.
And you can think of function in a different way.
For example, you think a function is just
a summation of many impulses.
So we know in the linear system theory,
continuous function can be expressed
as a convolution of the function and the delta,
Dirac delta function.
So the meaning of that convolution
is basically, say, arbitrary function,
you cut it into small pieces.
You add all these small pieces together,
all these right impulses.
Put together, that's just your original function.
That's one way to represent function.
The same function, and I explained to you,
in this case, a periodic function, continuous function.
Just think of one copy.
Other copy is the same thing.
This is one copy of a periodic function.
This function can be represented as a summation
of many sinusoidal functions.
These sinusoidal functions can change in frequency, phase,
amplitude.
And the trick is, really, you'll find the amplitude, the phase,
you'll find all these parameters so
that the sinusoidal components added together
will recover your original function.
So just verbally, these ideas are not challenging at all.
Just how function you represent in different ways.
One way, view the function as many, many pixels
impulses, delta functions, a kind of particle view.
The other way, you just decompose the function
into multiple wave components, as shown here.
So what we learned in the previous lecture,

nothing more than this slice, really.
It just say arbitrary function, you can represent it
as a DC component plus a bunch of cosine components
and the number of sine components.
And the components are different in terms
of amplitude, in terms of frequency,
because n can go from 1 to a very, very large number.
So the larger the n value is, the higher the frequency
will be.
Sine, cosine, the two components, sine components,
cosine components, for the same frequency,
you can combine into a single sinusoidal function
with a phase factor.
That's why I say phase difference.
But if you decompose into sine and cosine parts,
you don't need a phase.
So just a bunch of DC components,
cosine, sine components.
The trick is, how can you find this coefficient?
And the formulas are available for you to do the trick.
So to find a0, you basically do an integral with respect
to f of t.
So this is just a known function.
f of t is a known function.
You do the integral with respect to t, and you get a0.
Likewise, you do computation like this.
So it's just one slice.
If you got confused last lecture, you just say, OK,
here is a formula.
Heuristically, I want to represent the function f of t
as a summation of a sinusoidal component in terms of sine,
cosine, and also special case constant.
How can we find these components, these coefficients?
These are formulas.
So if you do not understand what's going on,
remember the formula.
Then I give you arbitrary continuous function f of t.
You plug into the formula.
You get real numbers for a0, a and b.
And then with this a, b coefficient,
you just plot these sinusoidal waves.
Then add them together.
Magically, you recover the original function f of t.
The larger and the higher accuracy,
and the capital N here can go to infinity.
So you have a very accurate representation.
So this is summary really what I told you.
I explained how you derive, how you view the formulas
to compute a coefficient.
I say that is really inner product.
Geometrically, it's a projection
of this infinitely many dimensional vector.
In this case, it's a function projected
onto one of the basis functions, either sine or cosine
or constant.
Because the constant sine, cosine,
with arbitrary integer number n.
And these functions together form
so-called also normal basis, not much different
from three dimensional picture.
You have arbitrary vector.
You can represent that vector in terms of its x, y, z
components.
That's a geometrical picture.

And the geometrical insight is so important.
If you got confused at this moment,
I really encourage you to understand in the review
process.
But the very bottom line, you see,
this is a way to represent the function in terms
of sinusoidal components.
And these are formulas to compute a coefficient.
If you understand this part, accept as it is.
So this is Fourier series in real form.
And I said this real form is kind of complicated.
You have three terms.
And in complex form, we have a more compact representation
of the same thing.
So complex form, real form, is just essentially equivalent.
And with complex form, we have convenience
to write a less number of letters.
So just the same thing.
And the cn is a coefficient that we
can compute in terms of inner product
between x-pronential basis function e to the power
minus 2 pi int.
And in my book, I always put i first.
Minus i, 2 pi, int, doesn't matter,
with the function to be represented.
So the Fourier series can be written in the complex form.
So far, we have been saying the periodic function
has a unit period over the interval 0, 1.
And for other intervals, like from 100 to 101,
the same thing.
You just keep repeating the fundamental picture over 0, 1.
When the period is not a unit, we
say that can be easily extended.
So this is the Fourier series for arbitrary function
with period capital T. So I make it symmetric.
So you think the function from the minus t divided by 2
go all the way up to positive capital T divided by 2.
So this is a function over there.
And we can express this function in terms
of Fourier series in complex form.
So this is a constant for nth element.
And we do inner product here.
And this is the complex harmonic component.
And 2 pi is still there.
n is there.
Then you have t over capital T. And t over capital T
kind of normalize lowercase t.
And this is the normalization factor.
Capital T, 1 over capital T. So the range
of this quantity is still from 0 to 1.
But in this case, it is really from minus 0.5
to positive 0.5.
The periodic function, you think is starting point
from negative 0.5 to 0.5.
Or you think the period start from 0 to 1.
That's just for convenience.
Whatever you view it, as long as you
have a whole interval of a total length of 1.
And then you have a basic picture.
And the other adjacent and continuously many intervals,
you just repeat the basic function of form.
So this is the formula.
So easier way.
So what we learned can be summarized in real form,

equivalently in complex form.
Or you don't want to remember too many.
Then just remember this one.
So when you make capital T equal to 1,
so that covers the complex form.
This is the unit interval case.
And you can just do some computation,
convert complex form back to the real form.
Real form is nice because we see the geometrical meanings
crystal clear.
And with complex form, you need some abstract thinking
in the complex space.
Anyway, so this is so much kind of review or warm up
on what we learned to prepare Fourier transform.
So by now, you may feel, why bother
we use Fourier transform?
So we have a function f of t.
And we talked about how to represent f of t
in terms of a bunch of sinusoidal components
or earlier as a series of delta or impulses
to represent the original signal.
You have different ways to view the same thing.
Usually means you have a better understanding.
You have a larger flexibility to do things smartly.
Oftentimes, you have a task to achieve.
Common sense, you want to find the easy way
to solve the problem.
You want to use a simple method to solve either simple
or complicated problems.
So the Fourier transformation or the delta function,
all these alternative representations
give you alternative ways to view the problem.
And sometimes, complicated problem in one domain
will look much simpler in another domain.
So like Roman numeric, if you write down Roman numeric
to try to do multiplication, that
will be very, very complicated.
But if you just write regular numbers
or you write in binary numbers, then the multiplication
can be very easy, either for yourself or for computer.
So the representation is very important.
So the Fourier series or Fourier transform
gives us a way to convert a problem into Fourier domain.
And as we will see later on, the Fourier representation
makes some difficult problems very easy.
So that's one thing.
Another thing, the original function
can be quite complicated.
And we decompose it into some simple form,
just view it as a collection of some very simple form
of basis functions.
So we divide the original function into small pieces.
And we know how to deal with single sinusoidal or impulse
function, like the convolution operation was derived
by using the delta impulse.
You know how a system responds to an impulse function.
Then an arbitrary function is decomposed
into a series of impulse functions.
Then you add system responses to individual impulse functions
together.
So we're talking about two rather important principles.
So you always pray for simple ways to solve a problem.
And when the problem is sizable and you
want to divide and conquer or divide the rule,

this is a very important strategy.
So Fourier analysis is something like that.
And we learned Fourier series.
Now we talk about Fourier transform
corresponding to our chapter four in the book
draft I shared with you.
I think when I got time sometime this month,
I will just fix a few typos and polish a little bit further.
I will update you with the next version of the foundation part.
Anyway, today we are going to talk about how can we derive
Fourier transform from Fourier series.
Then we talk about some fundamental interesting
properties of Fourier transform.
And then we talk about a high-dimensional expansion.
And in the high-dimensional case,
I gave you an example to show you
the trouble is really worthwhile.
Once you convert an arbitrary image function
into Fourier transform, you can do certain things,
like you can remove noise very easily.
So you will get a better understanding
of the utility of Fourier analysis.
So you showed these slides before.
This is a summary of what we learned
in the previous lecture.
Also starting point, we used these slides as the first step.
Then I just keep deriving so-called Fourier transform.
So this is what I just explained to you.
Arbitrary function is a periodic function.
And the fundamental period is over minus capital T divided
by 2 to the other positive half.
So you define the function over this interval.
Then this arbitrary function can be
represented as a bunch of sinusoidal functions
in a complex form.
So you have a very compact e to the power i to pi nt,
lowercase t over capital T. By Euler formula,
we know these sinusoidal components eventually
can be broken down to sine and the cosine functions,
certainly at a different frequency.
This n can go from minus infinity to positive infinity.
This is a formula under the bottom line
is how you can compute coefficient c sub n in terms
of inner product.
So f of t and this general term, complex harmonic component.
So this is summary all we learned in the previous
lecture, OK?
We can do this, OK?
This is cn.
This is how you compute cn.
OK, let's just insert this right-hand side into the cn
place.
What do you get is something like this, right?
OK, we just insert cn into the Fourier series
in the complex form.
We got this, OK?
So this is 1 over capital T. When
we talk about a unit period, we don't have this.
But when you have arbitrary interval with period T,
you need to do this averaging.
So here is a kind of normalization.
This is averaging over the whole interval.
So we move this 1 over capital T out.
So you see, this is a summation, OK, weighted by 1 over T.

And inside this bracket, you have inner product.
And this right-hand side, you notice,
and you keep computing inner product.
So this integral, I call it inner product,
because two things really multiply together.
You do pointwise matching.
You got the partial products, then add them together.
Really here is you are integrating them together.
So these are many, many inner product indexed by n.
So n keep going from negative to positive integers, OK?
So for many different n's, you keep doing this inner product.
And for each n, you really can think
you are trying to evaluate the value at a frequency
point n divided by capital T. So this is a frequency point.
And to understand that better, so let
me show you this picture.
So you think the frequency point n divided by T,
n keep changing from negative to positive.
So you have many, many values.
So each n give you one point, OK,
when n equal to 0 got this point, DC component.
Equal to 1, you get this one.
Minus 1, you get this one.
For arbitrary n, you get a frequency,
this create a frequency point at the green location.
n equal to n divided by T, OK?
So this is a frequency point.
And you just think the interval is from minus capital T
divided by 2 all the way to capital T divided by 2.
And we can think the T is very large, OK?
You can keep writing n become larger and larger.
So this single period will become so wide,
cover the whole u-axis.
And the u really signifies the frequency axis, OK?
This is talking about frequency.
Originally, we think Fourier components
as discretized frequency for periodic function.
But if the period of a periodic function becomes so large,
so the minus T over 2, the T is so large,
minus T over 2 will cross minus infinity.
Likewise, T divided by 2 will cross positive infinity.
So I'm trying to build a bridge from a periodic function
to non-periodic function by writing capital T approaches
infinity.
So that way, you really just deal
with non-periodic function.
Non-periodic function is nothing but a periodic function
with infinity along periodic.
So anything after infinity along, we don't care.
So that's a trick.
OK, see, this is a comment of the right-hand side
inner product at infinity mean a discrete frequency point, u
equal to n divided by T. And therefore,
sufficiently large T and all integer n,
the intervals for u is dense on the whole number axis.
This number axis is just the frequency axis.
So you're sampling the point very densely.
And you really put the sampling point over the whole u axis
when capital T is very big.
So you just sample the whole u axis.
What's the interval?
Interval between adjacent sampling point.
So this is n over capital T, n plus 1 divided by capital T.
So this increment delta u equal to 1 over T.

This is nice to know.
So the whole trick is really summarized on this slide.
So now we can write capital T becomes really big.
So this increment becomes very small.
And these variables can be reworded
in terms of continuous variable.
In other words, n divided by T can be replaced by u.
So this is shown here.
OK, just look at this slide carefully.
Then we know the essential point of this lecture
and how we derive the so-called Fourier transform,
continuous transform, how we can derive inverse Fourier
transform.
So just try to follow me here.
So this is what we got.
When we insert Cn, the expression for Cn,
in the place Cn, we move the 1 over T outside here.
That's not an issue.
And here, we write capital T very big,
because I explained we want to find a Fourier expression
for non-periodic function.
How we get a non-periodic function,
we write the period of the original periodic function
very large.
So we just make infinity here, because T is so big.
Likewise, you have minus infinity.
So here, e to the power minus i 2 pi, minus i 2 pi.
So n divided by T. And I argued n divided by T
is nothing but u.
So the summation becomes integral.
And the discrete point becomes a continuous variable.
So when n is very big, you have many, many sampling points.
So this n divided by T becomes u here.
We've got T. So I of T dt.
So you've got this part.
This is a continuous x present.
So it looks very nice.
And the other side, it's just a copy down here.
So here, we utilize the relationship
u equal to n divided by T. So here, u divided by T.
I mean, n divided by T.
So just call n divided by T u.
So you can just get u here.
So any u, then you can recover.
So because when T is very big, all those sampling points,
it really becomes closer and closer, OK?
This way.
And then we can go step further.
So we call this whole transformation, this x
present, as a function f hat u.
Because the integral is raised back to T.
So u is a parameter.
And the parameter has a frequency interpretation.
So this is a Fourier transform of f of T.
And you've got a function f hat of u.
So this is the coefficient of Fourier series.
Stay here.
Then you still do summation.
So you add all the Fourier components together.
Again, here, n divided by T, you call it u.
And the summation is converted into integral.
Because the summation happens here.
Many, many terms.
And each term is weighted by 1 over T.

And each functional value is value,
say, this arbitrary sampling point.
The functional value is here.
And this value is weighted by this small interval.
As T increases, and the delta u is
the interval between two sampling point becomes so small.
Therefore, this summation, really,
just each term is a functional value weighted by the interval.
And all these things added together
is nothing but an integral.
So the summation becomes an integral.
And the integrant is a Fourier transform
of the original function f of t, denoted as a hat f of u.
And the kernel here is e to the power i 2 pi u t.
Because n divided by T is u.
So you've got this one.
So you see, f of t can be expressed
as an integral of the spectrum of the original function.
So the kernel here without a minus sign.
But when you compute f hat of u, you have a minus sign.
Otherwise, they look the same.
So this is so-called forward and inverse Fourier transform.
So given a function f of t, you do integral like this.
You've got a continuous Fourier spectrum here.
So this is a forward Fourier transform.
So given your Fourier spectrum, continuous Fourier transform,
and you can do integral or transform again
to recover the original function f of t.
So this is an inverse Fourier transform.
Now we can deal with a non-periodic function f of t
in terms of Fourier spectrum.
We can go forward and backward.
So let me give you some picture to make the abstract formulation
a little more visible.
Suppose you have a function, rectangular gate function.
This pi of t is just like a gate, right?
So you have this gate function.
The amplitude is 1.
The total length of the interval is 1.
So area under the curve is 1.
Looks like a gate, and it's rectangular.
You can call it either way.
Pi of t, we will use this function quite often later on.
So this is a gate function.
This is a non-periodic function.
Based on what we learned in the previous lecture,
you do not know how to express this non-periodic function
as a summation of a sinusoidal component.
It doesn't matter.
You know how to do Fourier expression
for a periodic function.
So let's just make a periodic function
from the single gate function.
So we make a periodic period 8.
So you have multiple copies, infinitely many copies
of gate function.
Looks like what you have here.
This is a periodic function, and we
know how we can represent this continuous periodic function
as a summation of a sinusoidal component.
That's what Fourier series is used for.
You can just plug in the formulas I explained later on.
You've got a Fourier series expression.
The coefficient can be computed according to those formulas.

So you've got a coefficient for DC components
and for the first harmonic component
and the second component.
And in complex form, they are symmetric for real function.
So the Fourier series will be obtained
according to coefficients we computed by those formulas.
So this is the coefficient we will have, something like this.
And the trick I mentioned, to find non-periodic function,
to find the Fourier spectrum sinusoidal expression
for a non-periodic function.
And what we will do is just keep increasing
period of the period function.
So we make 8 becomes, say, 16, 32, 64,
larger and larger.
So for each given period t, no matter how large,
you can compute it according to Fourier series theory,
those formulas.
And what's the difference?
When period becomes larger and larger,
the interval between adjacent Fourier components
becomes less and less.
So you see as t becomes larger and larger,
so you got a denser and denser sampling picture
in the frequency domain.
What's the horizontal axis?
So the horizontal axis is u.
Remember I told you the horizontal axis is u,
defined as n divided by t.
t becomes larger and larger, this interval
gets smaller and smaller.
So when capital T approaches infinity,
so this Fourier spectrum really approach a limit
that is a continuous function, even a loop of this function.
So that's a geometrical picture of how
you derive continuous Fourier spectrum from discrete Fourier
spectrum.
Desecrate Fourier spectrum corresponds
to periodic functions.
And the continuous spectrum really
corresponds to non-periodic functions.
So the trick is to make capital T approaches infinity.
So this is just a geometrical picture.
So this is a one-slice summary.
So given a one-dimensional function f of t,
you can perform this Fourier analysis.
You got a Fourier spectrum.
This is a continuous spectrum, but you
can understand this spectrum as a limiting
case of periodic function.
So the geometrical picture ought to be kept in mind.
So you have a very clear picture in the case of Fourier series.
Now in the limiting case, you have
continuous Fourier spectrum for a continuous one-dimensional
function.
So this is a so-called forward transform.
And it's a part of this inward transform.
So you have Fourier spectrum, spectral representation
of the original function.
Then the original function can be recovered
by adding all these individual infinity many.
But each component is so tiny.
But you add all of them together.
That's not summation anymore.
And this is integral.

But integral, summation, basically the same idea.
You add all these Fourier components all together,
and then you recover the original function.
This is the way we desired.
So we want to represent the original function
as a sum of many waves.
Now, so many waves.
Each wave component is tiny.
But infinity many added together
will give us a definite result. That's your original function.
So for all these mathematical operations, meaningful.
And I underlined in the previous class.
So the periodic function ought to be square integrable
over the interval 0 to 1, or over interval 0 to capital T,
or minus capital T divided by 2 to positive capital T divided
by 2.
So all these operations should be convergent.
So the mathematical definitions and derivations
will be meaningful.
In this case, we are talking about non-periodic function.
So the similar assumption applies here.
That means f of t is non-periodic.
But if you do squared integral over the whole number of sides,
so the function of f of t, although non-periodic,
but the squared function should be integrable.
That means the integration of the squared function
should be some finite number.
If you think about a function like a constant function,
and you just do integral squared or not,
you've got an infinite number.
Then you have a divergent problem.
So mathematical operation no longer meaningful.
So this is a summary.
And this summary, and I would say,
try to precede the beauty behind this formulation.
You see, this is a function.
And this is a Fourier spectrum.
By inner product, this function really
projected to infinity many basis functions.
So you've got all the coefficients.
Just like you represent a 3D function in terms
of x, y, z components.
That's the same thing.
So you have this Fourier expression.
The unit vectors, really the basis functions,
sinusoidal functions, in this form.
So geometrically, the same idea.
Then once you have this expression,
and then you can recover the original function
by adding all these small x, y, z components together,
you get an original three-dimensional vector.
In this case, you add all these small sinusoidal functions.
This is coefficient for a particular sinusoidal function,
e to the power i 2 pi ut.
You add all these tiny components together.
You recover the original function, f of t.
So we say you can go back and forth
without information loss.
Then the f of t is equivalent to height f of u.
That's Fourier transform of original function.
The equivalency really means almost everywhere.
If you have piecewise continuous function,
at this continuous point, say you
have finitely many discontinuous points,

the Fourier expression, like this one,
will converge to average of left and right limiting value.
So the right and the left limit is not perfect recovery
of original function.
But if your original function is continuous,
then you have pointwise equivalency.
But otherwise, this equivalency should be preceded
as almost everywhere.
So for continuous point, you can recover the original function.
But now, if you are talking about original function
discontinuous at a certain point,
then the recovery is not exact.
Anyway, this is a pair of forward and the inverse
transforms, or simply called Fourier transform.
If you like, you can say this is a pair of transform.
So this is just a mathematical summary of Fourier transform.
And it's not serious.
Serious transform, what's the difference?
Serious talking about periodic function,
put it in sinusoidal expression.
And transform, we mean we want to express
our original functions in terms of continuous Fourier spectrum.
How you get this Fourier spectrum?
You perform so-called Fourier integral.
So this is the idea.
OK, so this part is the mathematical part.
And the next part, I give you some examples.
And I will just show you the x-plane properties
of Fourier transform.
Maybe we have 10 minutes rest, and then I
will finish the next part, OK?
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties
of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
So this is the x-plane properties of Fourier transform.
.
.
.
.
.
.
.
.
Let's continue.
So let me emphasize
a good understanding of linear system
Fourier analysis is so important.

And if you just look at appendix of
some imaging textbook, so you see those formulas
and you plug in and you can compute. That's not
the most important. What I have been trying
my best to explain, I really want you to visualize
what's going on so you know the ideas
behind all these computations.
So the essential part already covered summarized here.
We can represent a non-periodic function
in terms of its continuous Fourier spectrum.
And with the continuous Fourier spectrum
we can recover the original function.
So this is just a complex wave and weighted
by this coefficient. And certainly this coefficient is
small. I'm not saying the amplitude of this
f is small. Really this amplitude is weighted by
delta u, du. So this whole thing
is a small weighting factor. And this
integral is really just a sigma sign. Add all
these components together and remember the
cartoon and a few students hold
small blackboard. The waves added together recover the original
picture. So that is the illustration of the idea behind
Fourier transform forward and
inverse. So this inverse Fourier transform
is just the summation process to recover
the original function. So understand the idea.
Then you do the computation. Otherwise you just do computation.
You feel you're like a robotic, like a computer.
We are human beings. We want to visualize
things so that you can imagine new applications
and new results.
The second part, we give you some examples.
So if you understand the formulas I showed you
on the previous slides, the rest are just
somehow straightforward. So gate function.
The gate function, we already defined the gate function.
It's just a standard gate. And the area and the curve is one.
So what is the continuous Fourier spectrum of the gate
function? We kind of already showed you, but now
make the process formal. So the continuous
Fourier spectrum is nothing but the integral
that I derived for you in the first part.
So you do the integral, you have limits,
infinity, negative infinity, because we write
capital T, really big. So that's just infinity.
Then you have inner product, original function projected
onto this complex
basis function, indexed by continuous
variable u here. Can you see here? 2 pi ut.
For giving you,
you do this inner product once.
Inner product is nothing, just projection. You just do
computation. The example was chosen to make our life
easier. So this is the function of value 1 over this finite
interval. So Fourier transformation
becomes this expression. And you just
solve this equation a few steps, and you have
this sine pi u divided by pi u.
So this is for unit interval. And for general
interval, so with period
capital T, and you got a more general
sinusoidal
I mean this whole function

is called sinc function. You got a more general sinc function.
You can just try to find Fourier transform
for this more generalized gate function.
But for standard unit gate function, you
have these results, but you can verify. And this
function, sine pi u divided by pi u
is very important, not only for gate
function itself. And we will utilize the
Fourier transform of this function in the next lecture.
So let's just look at what is the functional form
of this result. This result has a name called
sinc function. It's not sine function. Sine function is just the
sinusoidal vibration. But this sinusoidal vibration
is divided by pi u. You're
getting larger and larger, so you can imagine the amplitude getting
smaller and smaller. But when u is
proportional to 0, and sine pi u proportional to 0,
pi u proportional to 0, then you use
your calculus 1. You find the limit using
something called the
Ropiotop rule. So you can find the limit is 1.
So this is the functional form of
sinc function, something like this.
So this is the first example.
What is the continuous Fourier spectrum
of a standard gate function?
And if you like, you can find a non-standard gate function. The period
is not 1. And what will be
the sinc function for that gate function
of period capital T?
This is the first example. Second example, so suppose the
non-periodic continuous function is a
triangular function or triangle function. This is a triangle function defined
this way. What is Fourier
transform? So we want to represent
this triangle function with many, many
one-dimensional waves. Add them together,
you can recover this triangle function.
And computation is not so much different
from what we explained.
We plug in the triangle function into
the formula for Fourier transformation. So you
have this inner product. So the integral from minus
infinity to infinity. This is a function. Then you have this
basis complex, basis function, e to the
power minus i 2 pi
2 pi.
In my book, I call it UT. But here, this example
I copied from the Stanford textbook. They call it
S. It's not a function of T. It's a function of X.
It's really variable. It doesn't matter. So in this case, you have
the frequency as S. So S,
the Fourier transform is performed with respect
to X. And just keep doing the trivial
computation step by step. And what you have is
sinc squared. Then this is U.
Here is S. So sinc squared. So that's just
somehow this. See, this is the
gate function. The Fourier transform is
a sinc function. So here is a sinc function.
And for this gate function, you got sinc
squared. So you have many examples. So any function
you just plug into the Fourier transformation.
And you just do the analytic computation.

And sometimes you are lucky. You can find
closed form solution. And many times, you cannot
find closed form analytic
solution. In that case, you can do numerical computation.
You can still find Fourier transform in terms of
numerical values. And a few more
examples shown here. They are nice. So you have this
Gaussian function. E to the power pi
t squared. So you have this bio-shaped Gaussian
function. And this function is certainly
square integrable. Then you can perform Fourier transform
upon this Gaussian function. The Gaussian function
is elegant. The Fourier transform, you plug
it into Fourier transform formula. You do the
analytic computation. It's
Fourier transform. It's still a Gaussian function. So in the
time domain, the variable is t. In the frequency domain,
the variable is u. And I use u to index
the frequency component. And if you use
sigma, or use s, or use what?
You check the textbook website and the different
versions of Fourier transforms are
available. But you really need to see what's going on. That's in the product
that you compute. And you have one
variable in one domain. Time or space or whatsoever.
And in the reciprocal space, you have frequency omega,
f, u, s whatsoever. But this is just a way
to represent the original signal in terms of
sinusoidal wave functions, different kinds of wave components
added together. And now let's look at
a very
ridiculous case. It's c. So c is
constant. And I have been saying to write
Fourier transformation, all the mathematical computations,
derivations, make sense. We ask
function, remember the IO2 space, that means
function, you do the integral, you perform the inner product,
we make sure the Fourier coefficient
converges to some finite number.
And the c constant, if c is not 0,
and this constant function is not square
integrable. So you do Fourier transformation
upon this kind of function, you have to be very careful
how you interpret this some
generalized Fourier transform. So in the
textbook I explained, we try to understand
how you perform Fourier transform for constant
non-periodic function. And for some generalized
function, delta u or delta x,
delta t, those things
are not easy to explain. But
indeed I gave some explanation in the
book chapters. And so if you are interested, you can read.
But here you can just understand as
just some rigorous mathematical derivation in the sense
of distribution. And you can have this relationship
constant, the Fourier transform is a delta function. The delta
function, the Fourier transform is a constant. So read
the chapter for detail. But mathematical regular
is not stressed here. And for delta function,
you perform a translation.
So by amount of a, so if a is equal to 0,
so the delta function Fourier spectrum is a constant.
So all the frequency components need to be added

together to make this delta function. But if you
have a shifted delta function, the Fourier spectrum
is shifted. So the
phase shifting factor depends on
frequency. So for higher frequency, you have a larger
phase shift. For lower frequency, you have a smaller
phase shift. And we will talk about
this phenomena in the next part about
properties of Fourier transform. So right now you're just
thinking these are a bunch of examples. Some are easier.
Like a gate function,
triangle function, Gaussian function, these functions are nice.
So you do integral, the functions are square
integrable. So you can understand the existence
and so on using your conventional mathematical sense.
But when you have some generalized function,
some function not square integrable, like constant
delta function, you need to use deeper
mathematics. And I explained a little bit in the
chapter, but I'm not going to explain it here.
So much for examples. If you like, you can find more
examples. Now we discuss properties
of Fourier transform. The Fourier transform
as I argued has been well motivated,
clearly defined. And what are the properties
of Fourier transform? They have a bunch of very nice properties.
So first, Fourier transform, you give an arbitrary
function f of t as an input through a box,
black box, called, it's not a black box, you should know
everything. It's a white box. Through this box or system
called Fourier transform, what's the output? Output
is Fourier spectrum of the
input function. Then we can ask
if this system or the transform is linear
or not. And we learned the linear system
theory, so any system we can always ask
this question. And the answer is positive. So
if you have function f
and function g, you perform Fourier transform. You got
Fourier transformation for f, Fourier transformation
for g, again, and a lot of versions of
Fourier transform. Here just copied from
Wikipedia. If you really prefer unified treatment,
read my chapter. Here just show you different things. So do not
stick with one notation, one set of
notations. So here, so the Fourier transformation
is really linear. So that means if you add
input function f and the input function g
together, or you linearly combine them together,
the combination, the Fourier transformation of the combined
f and g linearly combined version,
in the Fourier domain, you see
the same linear combination of their
corresponding Fourier spectrum. So translation
property, and I mentioned that this is the
previous slide for delta function. For arbitrary function,
so you have f
is the original function. Then you have its Fourier
transformation, height f, Fourier component sigma.
If you do translation in one domain, then you have
phase vector weighted upon the Fourier transform.
You have modulation. Modulation
seems just like the other way. You translate it in Fourier domain.
You go back with this weighting factor in the spatial

or temporal domain. And the scaling, so
you just say you have original function f of
t or f of x. You have Fourier transform.
Then what if you just scale f of t or f of
x with a factor a? Then is f of
ax, and what will be Fourier transform? And the
conjugate and so on. So all these are properties you can read.
Let's look into more detail. Okay, the first
linearity. So this is just, again, this part copied
from Stanford textbook. So
you have a function f and g.
There are Fourier transformations. So you have Fourier transformation.
You have this summation, the summation
of the original function. Then you perform Fourier transformation.
The result is the same as summation of
Fourier transformation of f and Fourier transformation
of g. So this is additivity.
So in the system, linear system lecture, we explain
that. How about the scaling or homogeneity?
So if f is scaled by alpha, then we perform
Fourier transform. That's the same as you perform
Fourier transform of f. Then you scale the
result by the same scaling factor. So you can verify
these two properties according to the formula
by defining the Fourier
transform. So you can just see here, just as an example,
you show additivity. And you can similarly show
the scaling property. And for
safety property, again, and all these
properties, you have the definition. You plug in the
translation or scaling or
whatever. You plug any modification
into the definition. Then you show what is
your outcome in terms of the Fourier transform
of original function. Then you find the relationship. You call it
the corresponding property. In this case, we are talking about
safety property. So you save the variable
by, save the f of t.
So the variable t by b. So you have
this. You perform
this substituting transform, u equal to
t minus b. So make sure this f
becomes f u. Then you just change the limit. You just
do step by step derivation. Then
you got this original Fourier transform. And
you have this phase factor, factorized
out. So you know if you have
original f of t corresponding to
Fourier transform capital F,
then if you do translation by b,
then you have, by b in the time domain, then you have
phase factor in the frequency domain. So these
derivations are not complicated.
So you can review just a way to
remember the property and also
a way to familiarize yourself with Fourier
transform. Scaling, I mentioned scaling. So scaling
property really needs to be discussed in two cases.
One case is a
greater than zero. The other is a less than zero.
If a equal to zero, that doesn't make sense because
originally you have a function f of t. So t will
change from minus infinity to
positive infinity. If you change

t to zero, that means you turn one dimensional
function into a single point. That doesn't make sense.
So we don't want a equal to zero.
So when a is greater than zero,
you do the derivation. You see the
original function, f, then you have the Fourier transform
capital F, or you just
call the height f, whatever notation you choose. So if
you make the function scaled in terms of
variable t, so t becomes a t, then the Fourier
transform will become 1 over a times
original function, but the original frequency
component becomes s over a. So see
the variable in time domain, a t, the Fourier
domain becomes s over a. So the factor
in one space is a, in the other space is a reciprocal
of the scaling factor. So this is
for a greater than zero. And for a
minus than zero, the mathematical reason is
between the lines. So the resultant Fourier transform
is minus 1 over a here, then still same
thing. So the difference is minus sign. Because you already said
a is negative, so minus 1 over a
is the same thing as 1 over
absolute value a. So summarizing two cases together,
so we say f t
has a Fourier transform capital
f of s. And in my book I call it
height f u. So you have a pair of Fourier transform
here. Then if you scale variable
p by factor a, then the Fourier
transform of original function will be modified
with scaling factor 1 over absolute
value a and original Fourier spectrum.
But the variable is weighted by 1
over a. And inside this
parenthesis, you do not have absolute sign. So this is
just scaling property. And again, I like
visualization. So visually, what does scaling
property mean? So look at this picture. This is
the top part is original
function.
Then bottom part is Fourier transform, color coded.
So for this blue triangle function, you have
this light blue Fourier spectrum. So the
blue triangle is kind of a sub. In the Fourier
domain, it's kind of spread out. So
when you use scaling factor to scale this function,
so you turn this blue function into
the same shape, but just the horizontal
axis or coordinate is deleted a little
bit. So you have a little fighter triangular
function, this orange function. Then the Fourier spectrum,
because in the T domain, the function got a little
fatter. And the
frequency spectrum got a little
bit slimmer. So you get narrower.
So you make the triangular function even
wide. So you got quite a sub peak
in the frequency domain. So the two domains,
they are really closely related. They have
duality. So you have very narrow in one space,
you will have very broad profile in the other
space. So if you have a delta function,

it's very narrow in one space. You have constant, very
flat, wide in the other space. And the
sweet spot is Gaussian function.
So you have Gaussian function in one domain, and you have Gaussian
function in the other domain. It can be the same.
So this is scaling property
and visualization of the property. And you have
many, many properties. So this is derivation. So you can ask
many questions, and you consider different operations
you could perform on the original function. Say you have
original function f of t. The Fourier transform
is height f of u. If you do derivative
operation upon original function
f of t, what will be the Fourier
transform of the derivative of the original function?
So things like this, you can keep asking many, many interesting
questions. And you can just plug in the fundamental operation,
in this case, derivation, into the
Fourier transform formula.
Then you do derivation step by step, and
you will have the relationship. Say the Fourier transform
of derivation of function f. What's a Fourier
transform? Original Fourier transform before
derivation is capital F of s.
So with derivation, you have this factor,
2 pi i s. So you have this phase vector,
as a weighting vector, upon original
Fourier transform. So you do derivation in
time domain. That is multiplication
in frequency domain. The multiplicative
factor is frequency dependent.
For different frequency, you have different value, and it's
linearly proportional to the frequency component.
So that's derivation.
Very important Fourier transformation pair
is called train of
impulse function. Because the train of impulse function
looks like a comb, also paired combs, we call.
So you have many delta functions with
period delta t. So the n equal to 0, you got
delta function at the system origin. n equal to 1,
so you got t minus delta t, so you have delta
function. Right safety by period
delta t. So you have many of these. So you have a
train of delta functions in time domain.
We will utilize this slice extensively
in the next lecture. The question here is, if you perform
Fourier transformation of this
comb function, a bunch of delta functions, a series
of impulses in the t domain, what is
the Fourier transformation, Fourier
transform of this time domain comb?
And again, it's just a comb, but with
different period.
The period in t domain is delta t. In the frequency
domain is 1 over delta t. This is
consistent to the scaling property. Imagine that I make
one thing, a factor in one domain, then the
corresponding original function of spectrum
will be a factor. So narrow in one
domain, factor in the other domain. Here the period
is smaller in one domain. The period becomes
larger in the other domain. Really just similar thing.
But here we are involving

delta functions, generalized functions. So the
relationship is shown here as a tool you can
utilize. But why we have this relationship? And when
we deal with delta functions, deal with functions that
is not square integrable, will the result still
mathematically meaningful? And that is what I mentioned involves
deeper mathematics. And they are rigorous, but I don't
have time to explain in detail. If you are interested
again, read the chapters. I try to make BME version
of the explanation in what sense these formulas
holds will not give you absurd results.
Next theorem is probably
most important property of the
Fourier transform. And also
very important property for linear system
theory is called the convolution theorem. And the convolution
we already learned is something, remember
example, you have one vector or one
function. You have a second function. You flip one, then you
match together, just add a parcel product
together. And you need to keep doing safety, get a bunch of numbers
and these numbers put in a list. That's the result of
you discrete convolution. And in continuous case, you
have two continuous functional forms. You flip one, then
you translate the functions continuously
over interval. And for each translation, you match
two functions together, that's multiplication.
The multiplication is integrated with respect
to the variable. And this whole thing, whole process
is called convolution. Why convolution is important? Because
linear system theory is important. Once you know
safety invariant linear system
has an impulse function, either
continuous impulse response function, either continuous
or discrete. Once you know that
for arbitrary input through convolution, you can find
output of the system. So from input to output
and that's a forward process that we do routinely in engineering
practice and in other fields, many other important
research fields as well.
So this is convolution. I also mentioned that convolution
is advanced form of multiplication. Now
we can make the statement very specific. Actually
convolution in T domain is multiplication
in Fourier domain. So they are equivalent.
So you can perform convolution as I told you
in your homework, in the class. You can continue to do convolution.
Or you don't do it. You just for the given
functions, f and g, you find
Fourier transform for f, Fourier transform for g, you multiply
them together. Just the standard multiplication
in the Fourier domain. This
product is Fourier transform of
convolution. So to find convolution, you can do multiplication
then you perform inverse Fourier transform.
You get exactly the same result. So this is
just
the geometric interpretation of
convolution theorem. So let me give you a visual example.
Remember the gate function. You do Fourier transform.
What do you have? Remember you got a single function.
Then I give you a second example, a triangle function. You can think
two gate functions. You do
convolution, two gate functions.

Convolution of two gate functions gives you a triangular function.
I just told you the gate function is a single function
in Fourier domain. So the convolution in
spatial domain, two gate, you got a triangle. So in Fourier domain,
the two gate functions are two single functions. So you multiply them
together. A single function squared, that is Fourier
transformation of the triangle function. That is what you show.
So here you know why. Because this is
convolution theorem. So why do we have
convolution theorem? Just plug in
convolution operation in terms of
the definition.
The two functions, F and G, you do multiplication together.
You just plug in the definition
of Fourier transform. You keep doing things, make it
in the form of convolution. So you can go from
multiplication in one domain to the
convolution.
To this form, the convolution in time
domain or space domain, then you perform Fourier transformation
of the convolution result. That is nothing but
multiplication of Fourier transform. So just
the derivations, step by step, and then you
can review after class, put a cup of
coffee or tea, and just go through. So this is
derivation of why you can prove. Let me give you
some deeper insight. You may or may not understand, but this
is the way researchers are encouraged
to do. You try to understand those
trivial mathematics with some visualized
picture. So why you have this? So you can actually think
verbally. It's just my understanding. You cannot find
from the internet. Let me make some comments. So convolution
is a result
we derived, a process we derived for
safe environment linear system. So I believe
you can show for safe environment linear
system. So you gave the system a
sinusoidal input. Then what will be output? The output
will be also sinusoidal. And also for
safe environment linear system, you gave sinusoidal input
to the system. The output must be sinusoidal
or sinusoidal function at the same frequency.
So linear environment system would not give you
anything, say, higher frequency, lower frequency,
and it will not change the form of input function
if the input is sinusoidal. This can be shown.
So in other words, it says convolution
in T domain must be multiplication in Fourier
domain. So I jump a few steps. So think
about this. You have a sinusoidal input to linear
system. The output is still sinusoidal, but it may change
some factor scaled upon the original thing.
So there must be some specific number at that frequency
multiplied to give you output for this
sinusoidal function. So for arbitrary function,
you decompose into many, many sinusoidal function. Add it together.
Then put it through the system. For each particular frequency,
you just weight the frequency
with a frequency-specific number. So that is
multiplication in Fourier domain.
And that particular frequency-dependent
number must be Fourier transform of
your impulse function. So the HT, you do Fourier

transformation. You got a Fourier spectrum with all
the coefficients or weighting vectors, frequency-specific.
So why convolution is multiplication
in Fourier domain? Just this sentence. So you understand
what's going on. And so I made a claim. So you have
sinusoidal function at a certain frequency.
You input into linear system, safety environment linear system.
The output is
invariability. So the certain
functional form put into the system, the output is still
same form of the function. And the only sinusoidal has
this property. You have delta function is delta. Output
generally is not delta. It's something like this in your homework.
So only sinusoidal has this property. So if you show,
I think you can show this. If you show this, you can
make further statement. So you think
about convolution in one domain. And you can ask this question.
Convolution in one domain. And in what domain?
You can think about Fourier transform, wavelet transform,
Hadamard transform, many, many transform. Each transform is just
different way. Orthogonal presentation is not unique.
Because also normal basis is unique. You have infinitely
many transforms. And any other transform will
also have this convolution theorem. The answer is no.
Because only sinusoidal function has this safe
shape invariability. Does not change shape.
Does not change frequency. And the second paragraph
really leads to an important conclusion.
And the convolution theorem only
holds for Fourier transform. If you have some X transform,
it's not a Fourier transform. You do not have convolution
theorem. So this is something you do Google search. You say
characterization of convolution theorem. There are multiple
mathematical papers talking about this characterization. Indeed,
the convolution theorem only holds for Fourier transform.
Why? You read pages, pages, mathematical derivation.
But really just visualize it. Just these few
comments I have. If any of you interested in mathematics,
any of you, I do not expect all of you understand.
Any of you really understand my comments. You want to write
a technical report, a small paper, just so that you can discuss
with me. This is a heuristic view.
View of convolution theorem.
You have next important property.
Passive identity.
This is, again, I copied from Stanford textbook.
I have a very good textbook. I used
this student in the previous semester.
Now I just have my book draft
so I didn't make things too complicated. Because that
textbook is quite safe. It's for whole-sized Fourier analysis.
This is the identity.
In T domain, squared
function, you just integrate them together.
What is this? I keep saying
to make classic mathematical sense,
function f of t must be square integrable.
If not integrable, you do not have a finite number.
You got divergent behavior, not easy to deal with.
Not to say impossible, but
much harder. You need to use deeper mathematics, like distribution
and so on in some mathematical sense.
But if you have this converging behavior,
much easier to understand.

Alternating current, the light,
what's the power? Power is just that you have oscillating
curve in the sinusoidal waveform
or any waveform. So it's just current squared.
That is, suppose you have unit resistance.
This is the measure of energy.
You do this integral, left-hand side is just total energy.
This is total energy in time domain.
The right-hand side, you have frequency components.
Suppose the waveform, the current, is arbitrary.
You can treat that as summation of many sinusoidal
OC circuit or OC current, but at a different frequency.
But for each given frequency, you have AC
alternating current. What's the power? For a given frequency,
the power is proportional to the amplitude squared.
This is the total power you see
in the time domain. So this is the straightforward high school
circuit teaching. Tell you for current
at a given time, squared times R, that's power.
For different time, you add all these together. That's total energy
the resistor consumes. In terms of Fourier analysis,
you think that arbitrary waveform is really
a number of many, many sinusoidal components.
For each component, the energy is amplitude squared
and all these sinusoidal AC current
added together, the frequency expression.
And this property basically says energy is
conserved. That's clear. That's the
physical perspective. What's the geometrical perspective?
So I told you earlier, I tried to encourage you
to think abstractly. So a vector, n-dimensional
vector, is a point in an n-dimensional space.
Then we say the function f of t is a vector, just an infinity
dimensional vector. And you use a vector to
approximate a continuous function, f of t. So f of t
is a vector, infinity dimensional vector,
is a point in infinity
dimensional space. And what is the length? What is the length
of that vector or that function?
The length is computed by squares, individual
components. Individual components squared, then you
add them together. You do square root. So you put square root
so the left-hand side is the total length of
vector or function f of t. So that's
the total length. And the Fourier transform, I say, is also normal
transformation. So you just think this f of t
in our original coordinate system is f of t.
And you put it in Fourier space.
Fourier space is really also normal transformation.
It's a coordinate system rotation. But this is just not
as easy as xy rotation. You rotate it to t and s.
But the geometrical idea is the same. It's just the
infinity dimensional space. You do a rotation called Fourier
transformation. Then in the Fourier domain, this
function becomes f hat s.
So each component you squared, then added together.
What's the geometrical meaning of right-hand side? If you put square root
here, that's the length of the vector in the
Fourier space. So geometrical meaning is that
the total length doesn't change. So this is
what I explained here. So you have geometrical understanding
why you have this identity. So certainly
you can just go through the formula. And the formulas
make things rigorous. And we like doing derivation.

But I always think for complicated things,
if you understand, just you can visualize things.
And that is better, higher capability
than you can just follow the derivation.
Oftentimes you follow derivation, but you really don't know what's going on.
So the geometrical understanding, you can visualize
the process. And that is the place,
that is the time you are getting
better understanding. So far we have
been talking about one-dimensional
transformation. And you can extend
Fourier transformation from one-dimensional to
two-dimensional. So the wave component is no longer
one-dimensional wave, rather just two-dimensional wave along
different directions. You can go
along x, along y, diagonally, whatsoever.
So this is a picture, very noisy
picture. You perform Fourier transformation, got this
two-dimensional Fourier spectrum. We know high frequency components.
Those wave oscillate at high frequency.
Oftentimes not very useful. Those components are noise.
So in Fourier domain, we just zero out all these
things. You see, perform inverse Fourier transformation, you get
noise removed. So this is why
say in Fourier domain, sometimes it's easier. And
this low pass, high pass filter, original image, you can
use low frequency components like this. And you can just
zero out low frequency components, keep a certain range
of high frequency components. That gives you edges
of the building. So that's another example.
And for rectangular, 2D rectangular,
2D impulse function, you can do analytic
computation. And in high-dimensional
space, you have something unique
called rotation property. Rotation property along
one-dimensional direction, you cannot rotate. You can flip. You cannot rotate.
But two, three-dimensional, you start having rotation property.
So this is a new thing, dimensionality dependent.
What's rotation property? Basically it says you have
2D function here. You have 2D Fourier spectrum here.
If you rotate the 2D function this way, the Fourier spectrum
will be rotated the same way, by same amount of
angular change. And why you have this
thing in any dimensional space? So this is mathematics.
You can follow through. Not very heuristic.
Let me give you a heuristic explanation. So you have
a function. You have Fourier spectrum.
So Fourier spectrum, each point is a wave
component moving in certain direction, just the wave.
So all these wave components added together
give you the original function. Now you rotate the original function
by a certain angle. And all the original
wave components are now going this way, the function like this.
The original function rotated by that direction. So these wave
components, to approximate that function, should be rotated by
same angle. So this is geometrical picture. Why you
have this rotation property of Fourier transform?
And you can just follow through the mathematics
to be sure. But this is geometrical picture.
So I hope you start understanding
the idea I'm trying to convey.
The foundational part is very important. And Fourier analysis
and linear system are not only important
for imaging, for engineering. Physics and mathematics

mainly utilize these tools.
I'm trying to give you some unique insight and understanding.
I particularly underline
the geometrical interpretation.
So far we show a function can be
represented as a delta function, or
wave function. So this happens to be what I call
the duality of information. So you can view
an image as a collection of pixels or voxels.
Or you view a picture as a summation of minisoidal functions.
So let's summarize what we learned. Maybe you like
this logo. So delta shows the particle nature.
This sinusoidal complex
thing is just sine, cosine, and wave. So these things
play together pretty much cover the linear system
convolution, Fourier series, Fourier transform.
And homework and TA will upload
so much for today.
Some students say they got confused about Fourier
series. If you are in that category, talk to me now.
...
...
...
...
...
...
...
...
...
...
...
...
...
...